The wave equation

The wave equation with zero ends boundary conditions models the motion of a (perfectly elastic) guitar string of length L:

$$\begin{cases} \alpha^2 \frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2} \\ w(0,t) = w(L,t) = 0. \end{cases}$$

Here $w(x,t)$ denotes the displacement from rest of a point x on the string at time t. The initial displacement $f(x)$ and initial velocity $g(x)$ at specified by the equations

$$w(x,0) = f(x), \qquad w_t(x,0) = g(x).$$

Method:

- Find the sine series of $f(x)$ and $g(x)$:

$$f(x) \sim \sum_{n=1}^{\infty} b_n(f) \sin(\frac{n\pi x}{L}), \qquad g(x) \sim \sum_{n=1}^{\infty} b_n(g) \sin(\frac{n\pi x}{L}).$$

- The solution is

$$w(x,t) = \sum_{n=1}^{\infty} (b_n(f) \cos(\frac{\alpha n\pi t}{L}) + \frac{Lb_n(g)}{n\pi\alpha} \sin(\frac{\alpha n\pi t}{L})) \sin(\frac{n\pi x}{L}).$$

Example: Let $\alpha = 1$, let

$$f(x) = \begin{cases} -1, & 0 \le t \le \pi/2, \\ 2, & \pi/2 < t < \pi. \end{cases}$$

and let $g(x) = 0$. Then $L = \pi$, $b_n(g) = 0$, and

$$b_n(f) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx = -2 \frac{2 \cos(n\pi) - 3 \cos(1/2\, n\pi) + 1}{n}.$$

Thus

$$f(x) \sim b_1(f) \sin(x) + b_2(f) \sin(2x) + \ldots = \frac{2}{\pi} \sin(x) - \frac{6}{\pi} \sin(2x) + \frac{2}{3\pi} \sin(3x) + \ldots.$$

The function $f(x)$, and some of the partial sums of its sine series, looks like

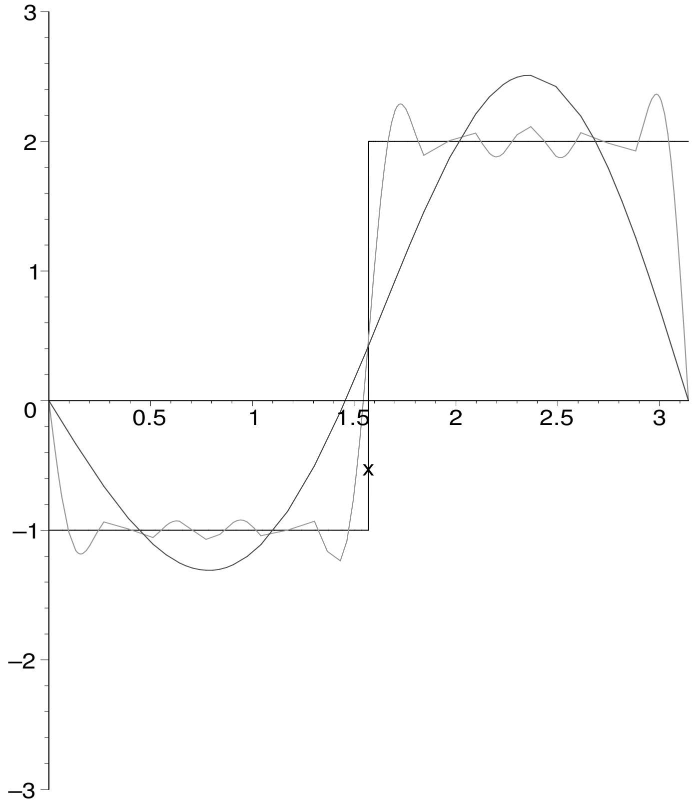

Figure 1: $f(x)$ and two sine series approximations.

As you can see, taking more and more terms gives functions which better and better approximate $f(x)$.

The solution to the wave equation, therefore, is

$$w(x,t) = \sum_{n=1}^{\infty} (b_n(f)\cos(\frac{n\pi t}{L}) + \frac{Lb_n(g)}{n\pi}\sin(\frac{n\pi t}{L}))\sin(\frac{n\pi x}{L}).$$

Taking only the first 30 terms of this series, the graph of the solution at $t = 0$, $t = 0.5$, looks approximately like:

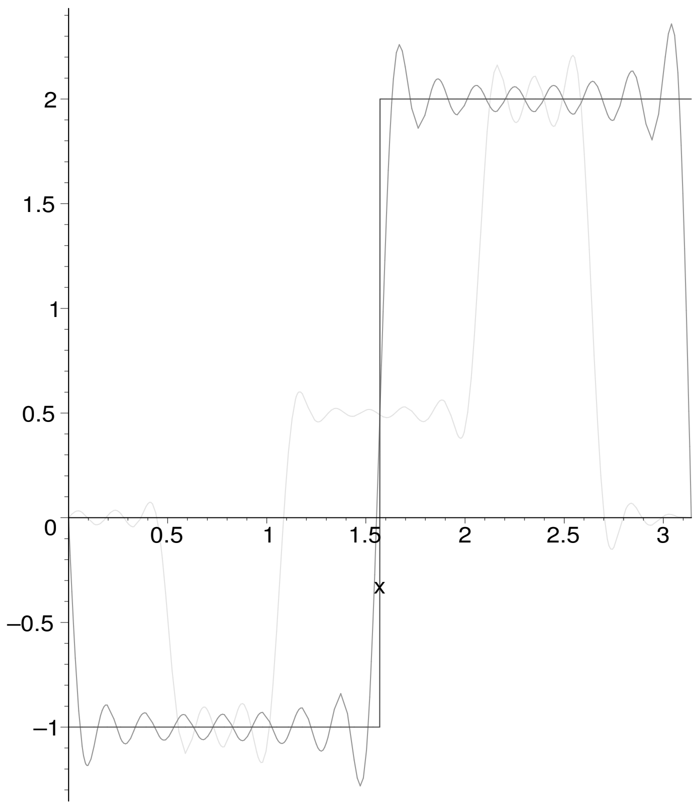

Figure 2: $f(x)$, and $u(x,0)$, $u(x,.5)$ using 30 terms of the sine series.